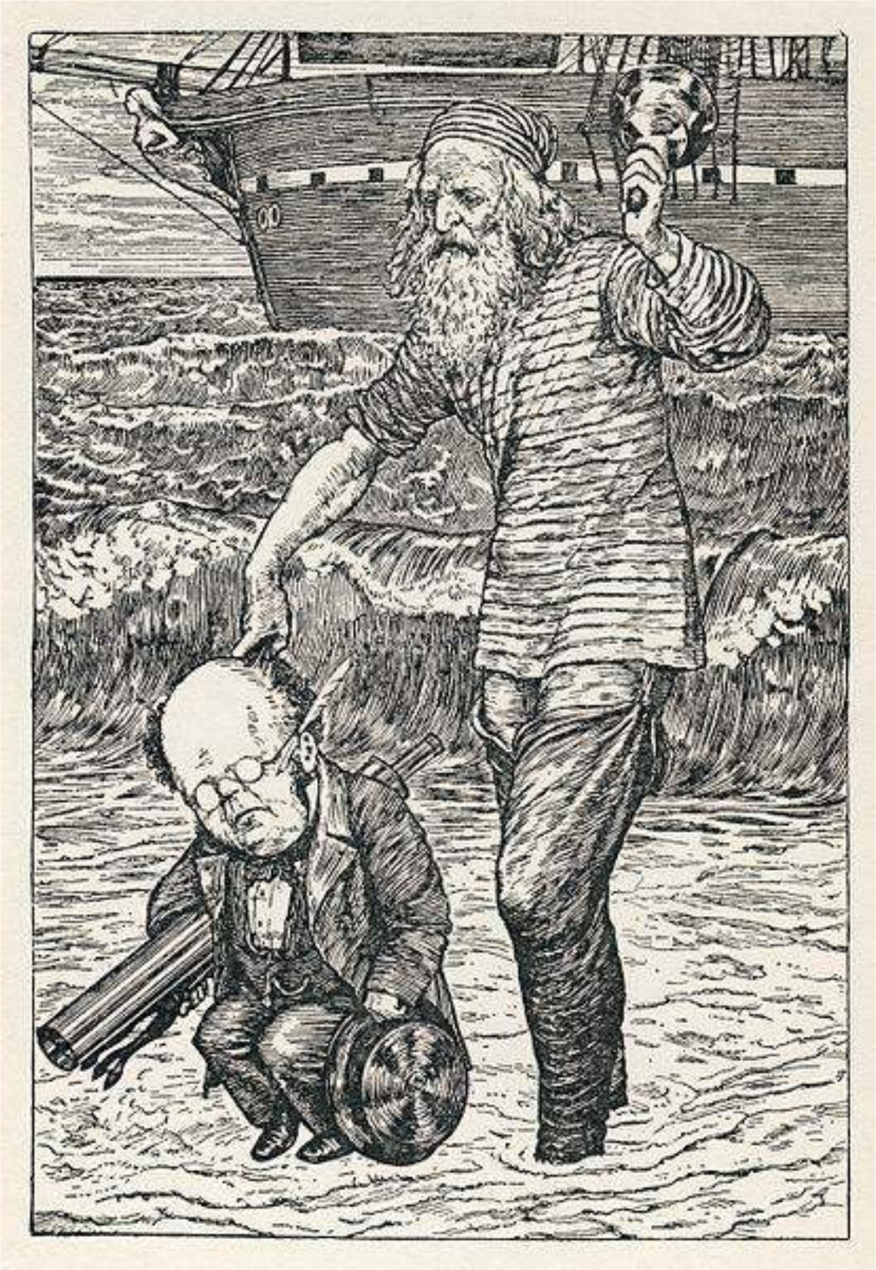
More Triple Proofs

What I tell you three times is true – Lewis Carroll

David McQuillan



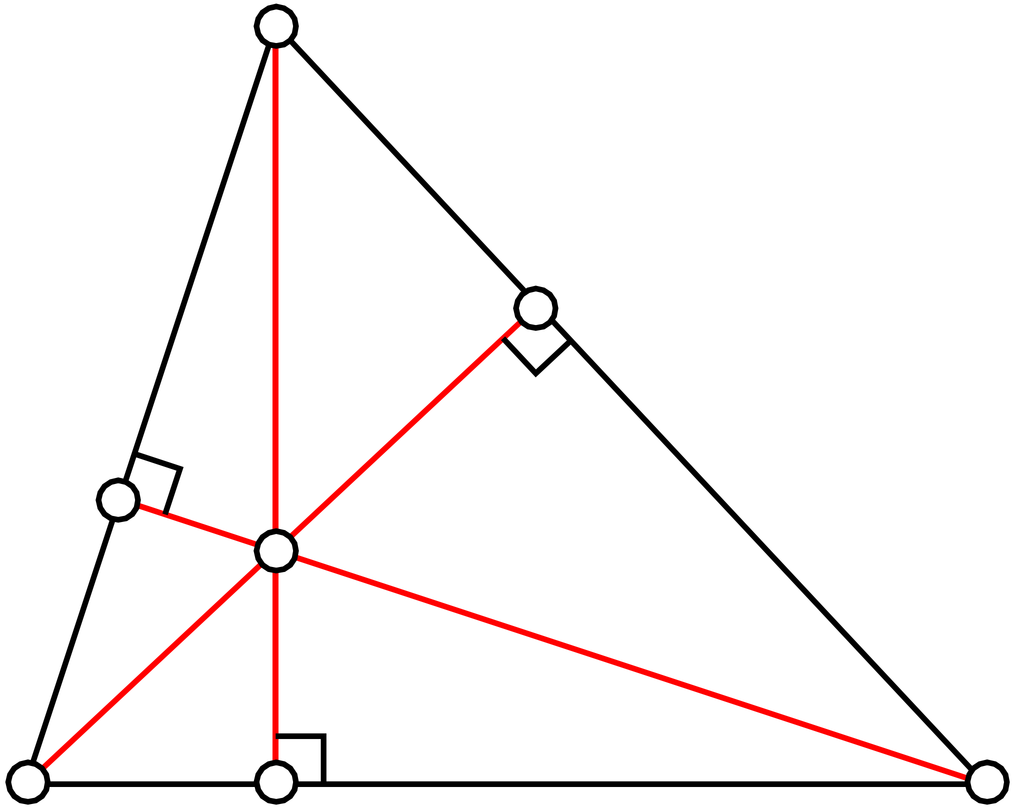
The altitudes of a triangle are concurrent

Euclid didn’t prove this!

It is not in the Elements.

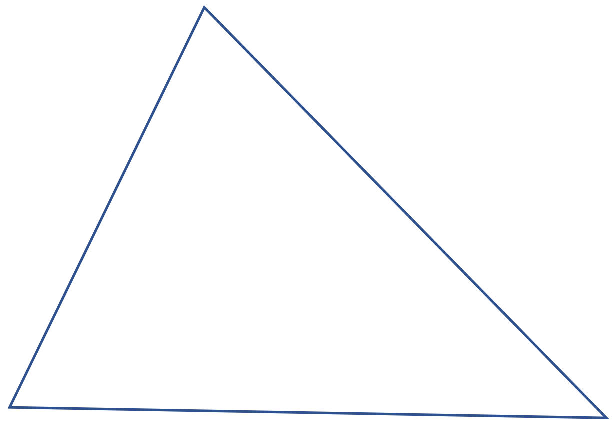
First known proof is by William Chappie in 1749

https://www.cut-the-knot.org/triangle/Chapple.shtml



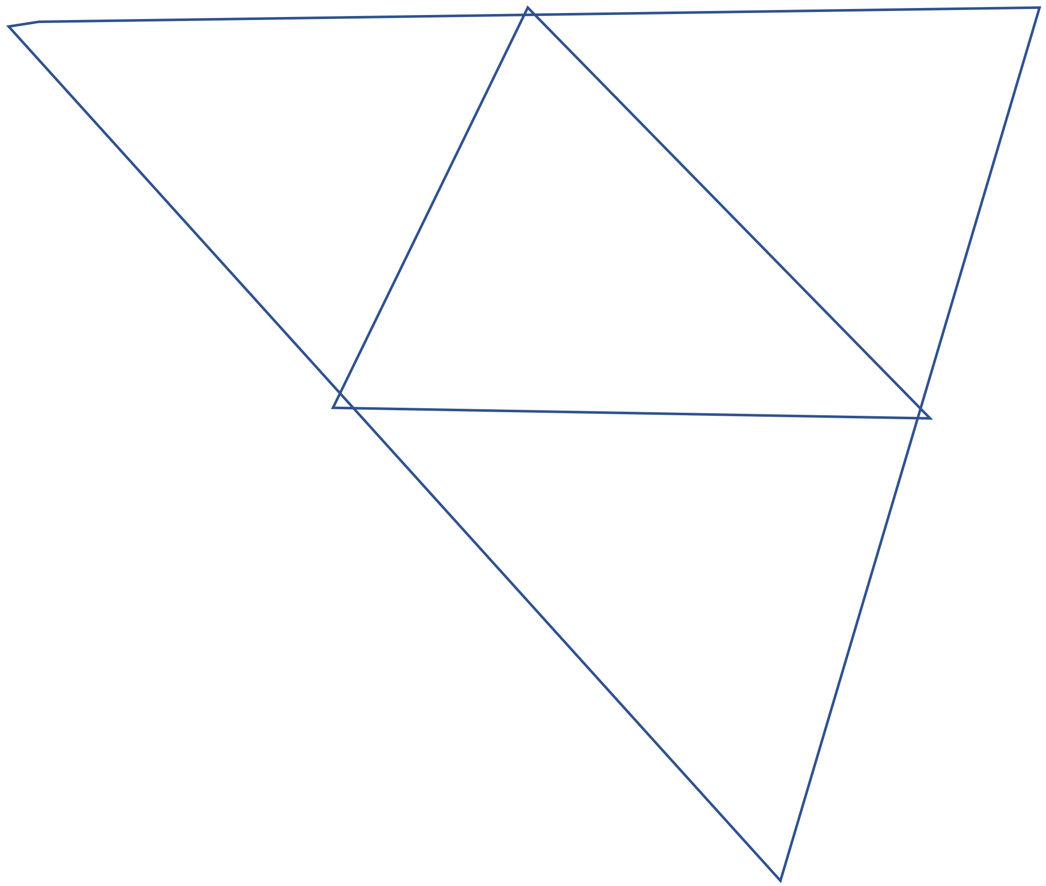
Proof 1

Start with the triangle



Proof 1

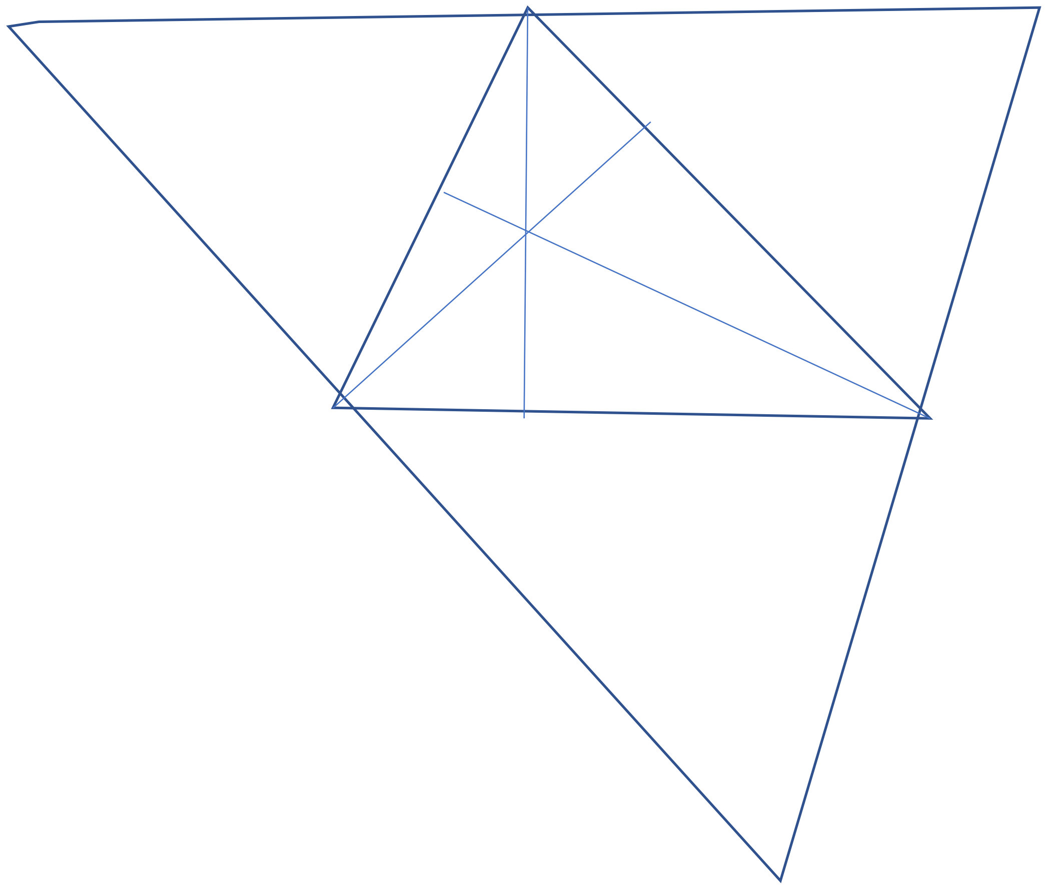
Draw lines parallel to the opposite sides so four equal triangles



Proof 1

Using that perpendicular bisectors of the sides of the big triangle are concurrent -

shows the altitudes of the small triangle are concurrent



Proof 2

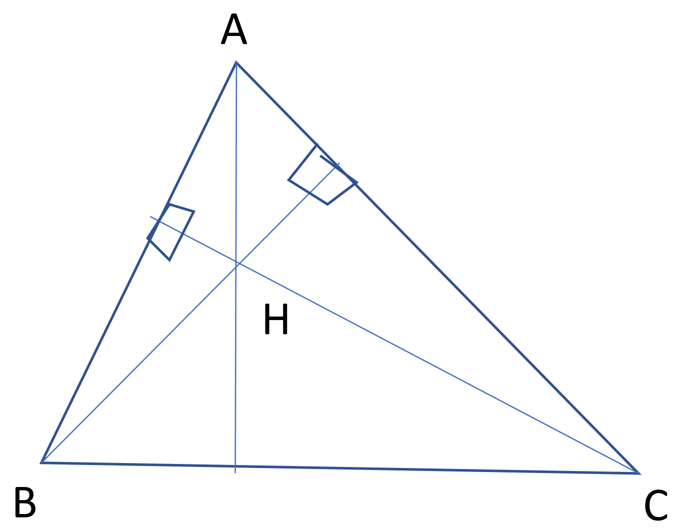
Given

BH perpendicular to AC

CH perpendicular to AB

Show

AH is perpendicular to BC



Proof 2

If A B C H are considered vectors from a a single point then using dot product

(H-B).(C-A) =0

(H-C).(B-A) =0

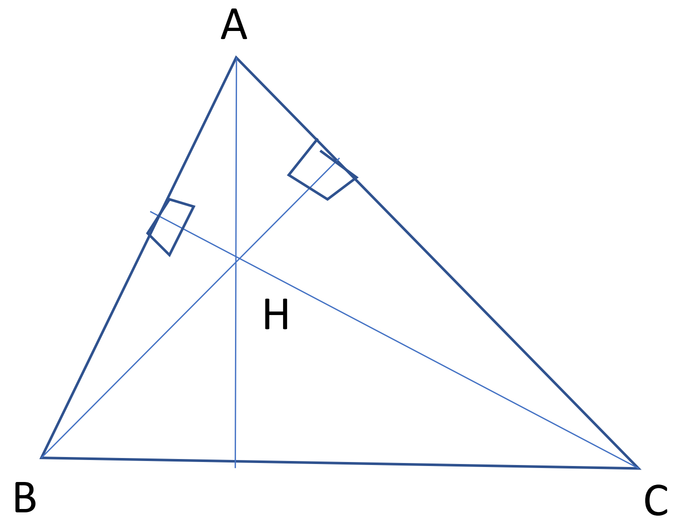
Expanding

H.C - H.A - B.C + B.A = 0

H.B - H.A - C.B + C.A = 0

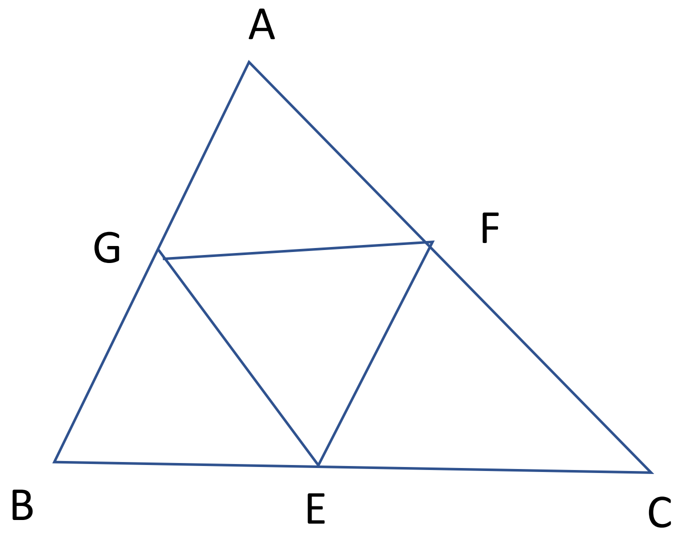
Subtracting

H.C - H.B + B.A - C.A = (H - A).(C - B) = 0



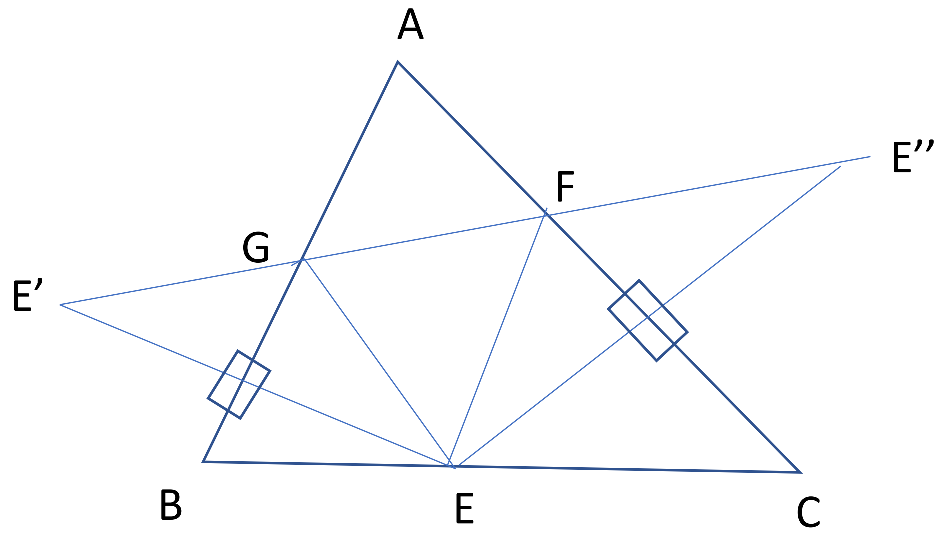
Proof 3

Start with Fagnano's problem – Find the inscribed triangle EFG of minimum perimeter.



Proof 3

For a given point E on BC the shortest EFG is given by length of reflections of E in AB and AC



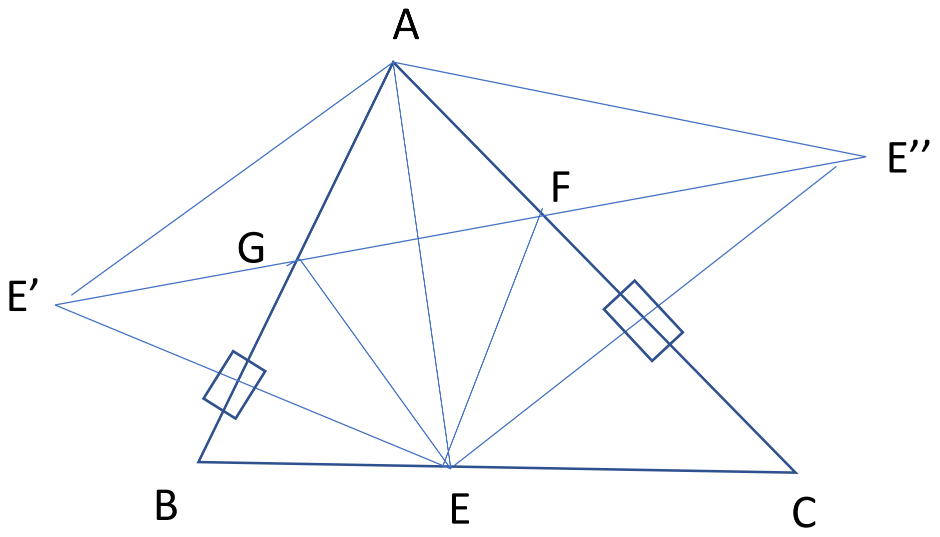
Proof 3

Draw AE’, AE, AE’’ they’re all equal.

Angle E’AE’’ is always double BAC.

So smallest AE’E’’ is given by smallest AE – i.e. when AE is perpendicular to BC.

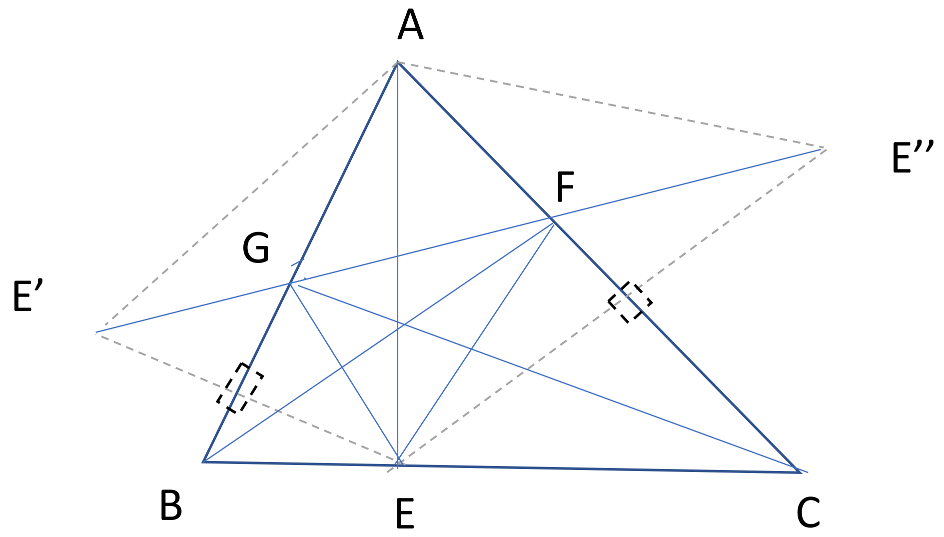
And that gives smallest E’E’’



Proof 3

But AFG = CFE’’ = CFE and similarly round the triangle. So if AE is perpendicular to BC then AE bisects angle GEF and same for the other angles.

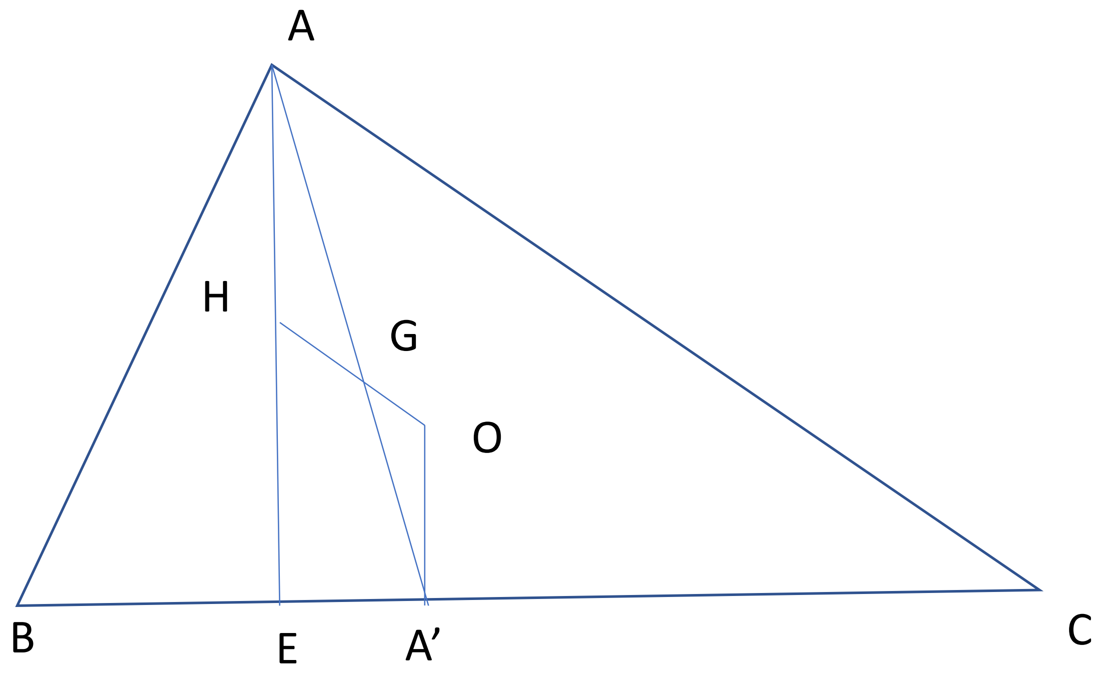
So the altitudes of ABC intersect when the angle bisectors of EFG intersect



And a Fourth for Luck

Euler showed in 1765 the circumcentre O, centroid G and orthocentre H are collinear and HG =2GO.

AG = 2GA’ and the angles are the same in AHG, A’OG



Folding a square of paper in three

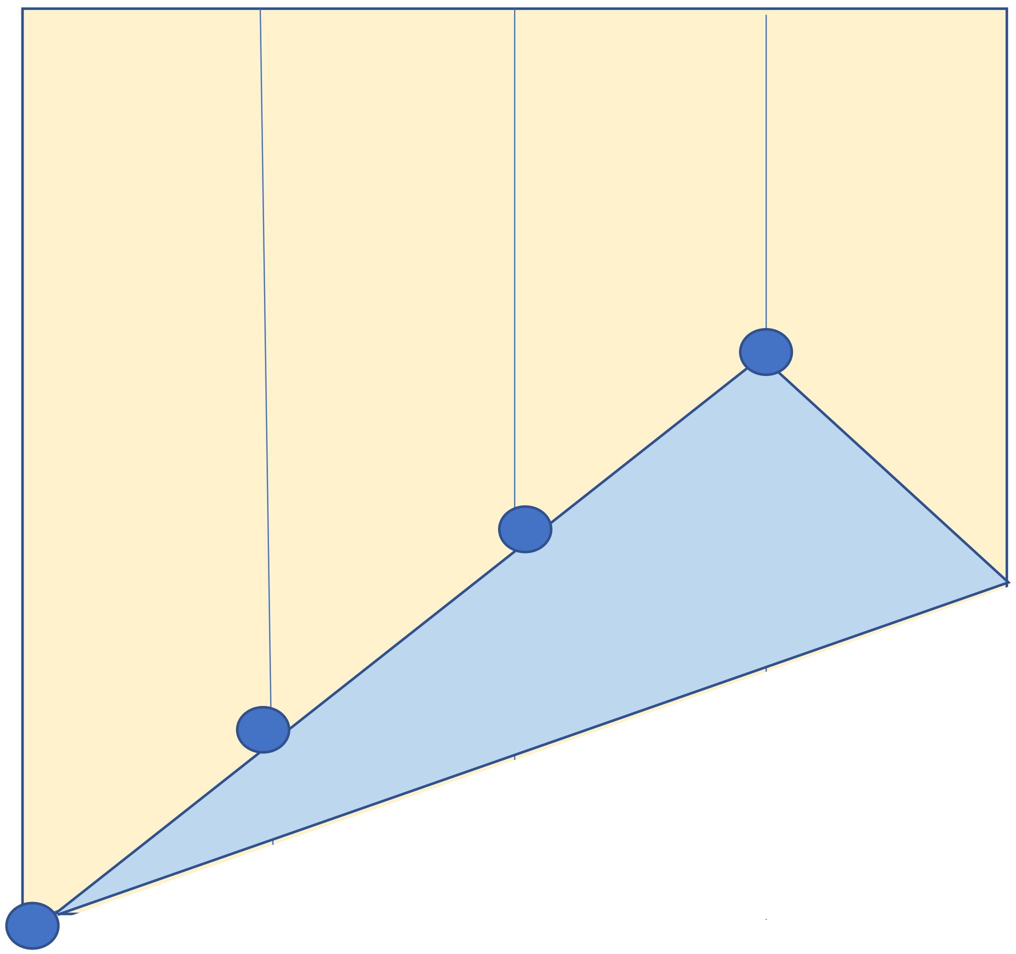
In fact only the last method depends on it being square.

It’s got to be mathematically accurate – not just, or even necessarily actually, accurate!

Using Proportions

For 1/N just start by dividing into $1/2^K$ greater than N.

Called the template method in origami – one can use two pieces of paper.

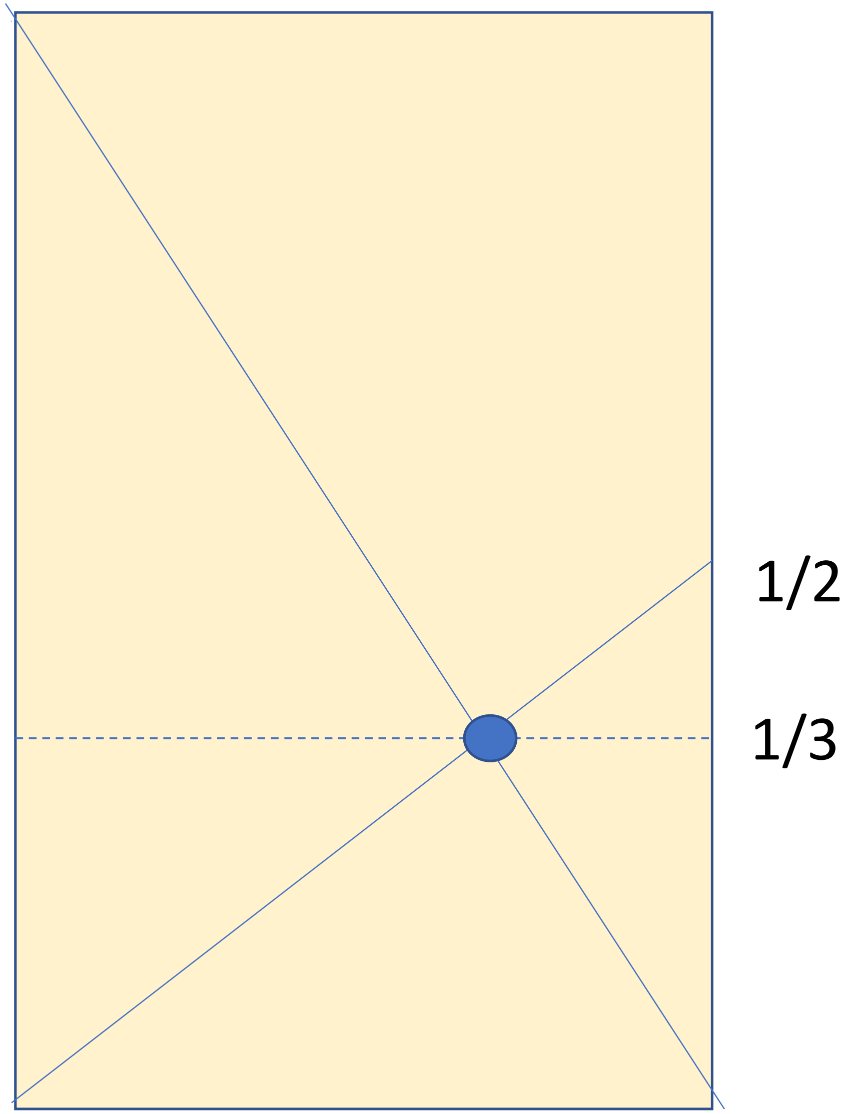


Intersecting Diagonals

1/N -> 1/(N+1)

Here

1/2 -> 1/3

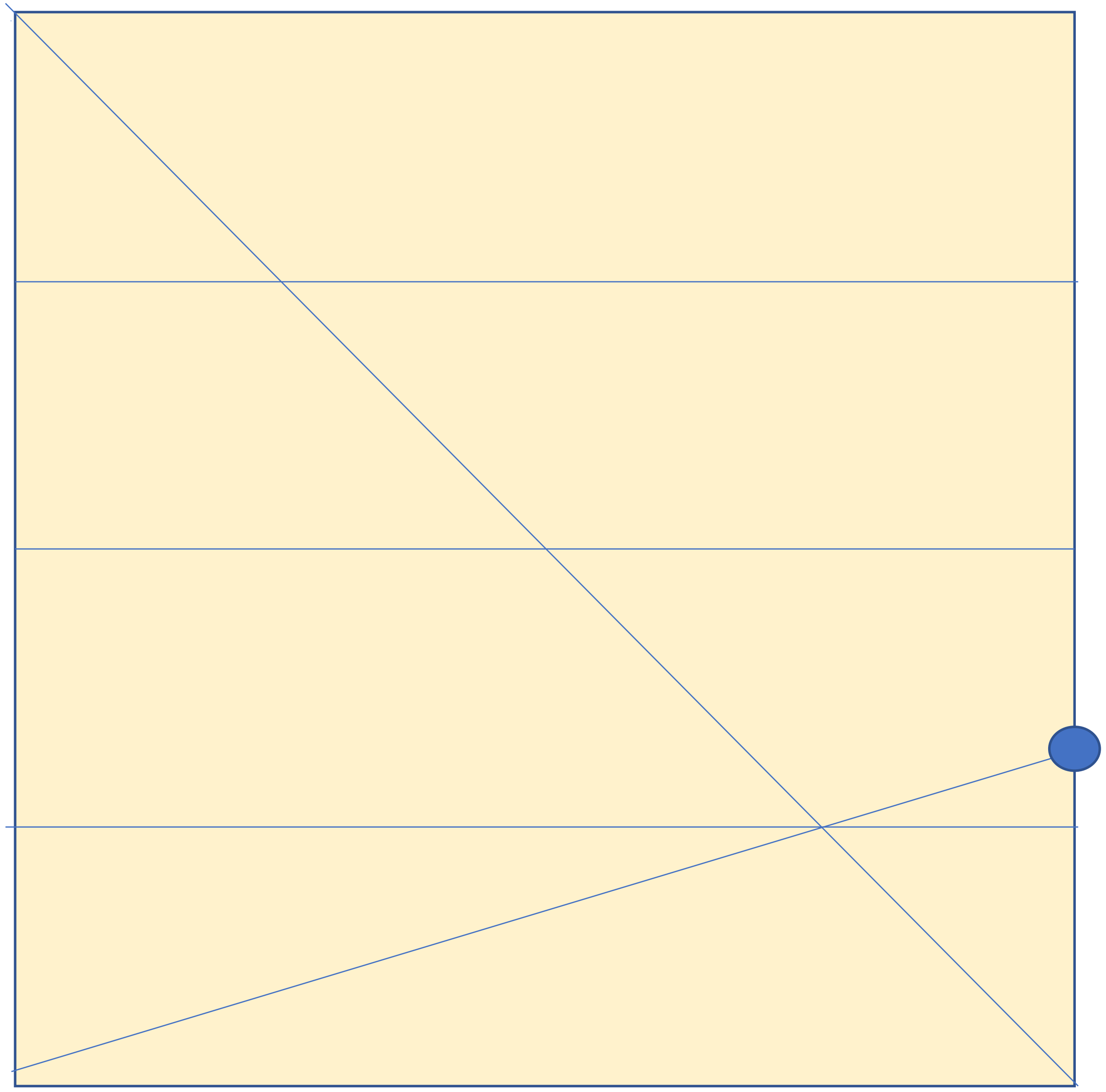


A variant

Let's not count this or we'll have four ways not three!

1/N -> 1/(N-1)

1/4 -> 1/3



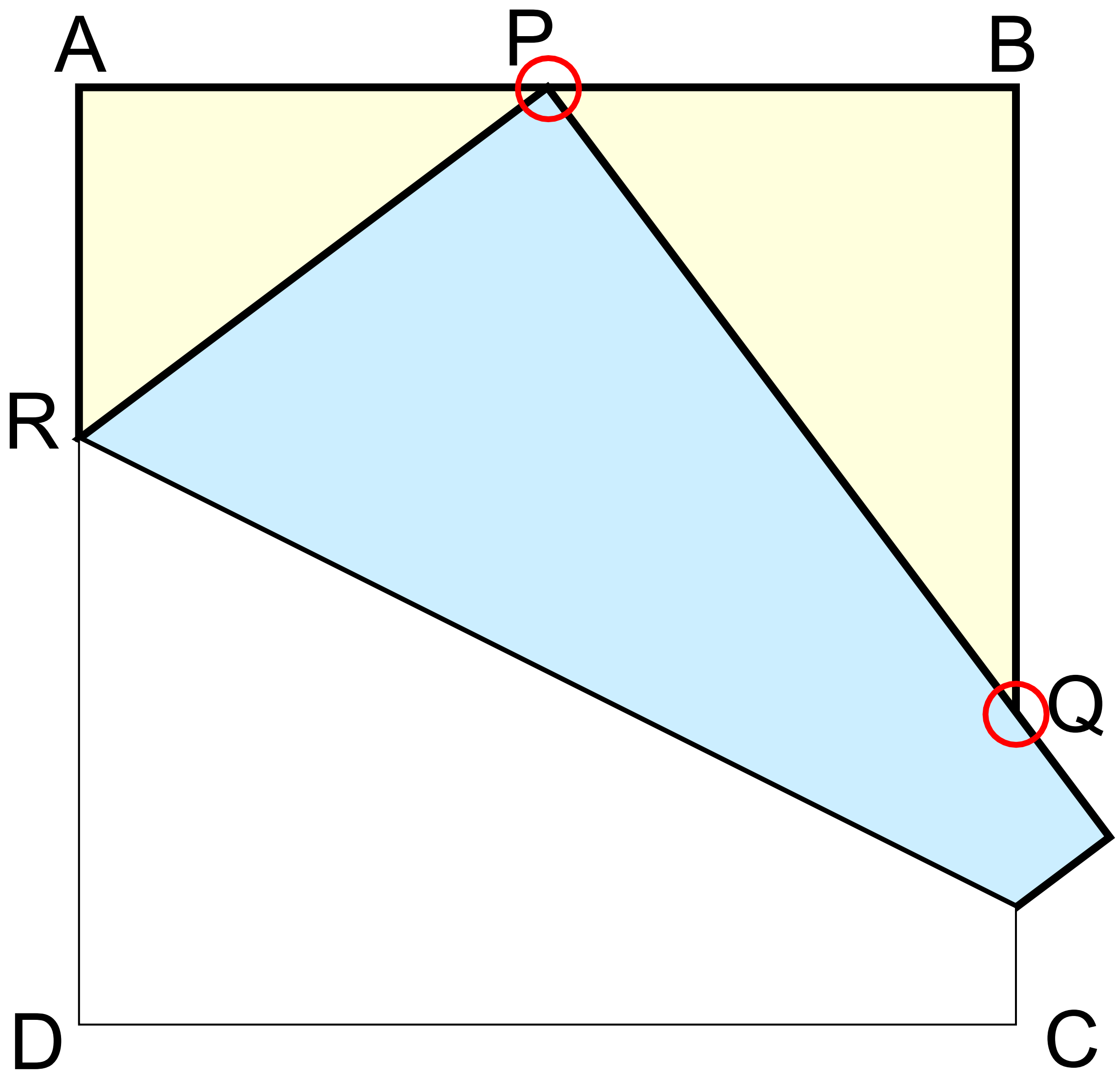
Using Haga's Theorem

If side of square is 1

Then

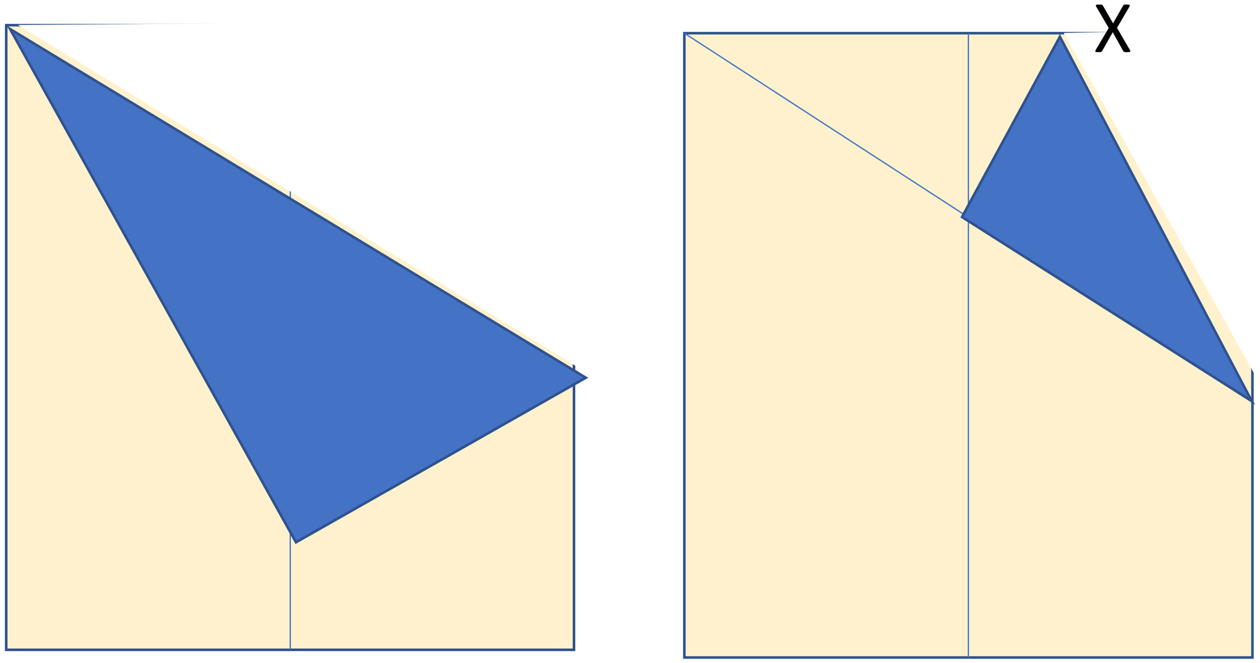
BQ = 2AP/(1+AP)

For AP = 1/2   BQ = 2/3



And another for luck from the talk

Using 30° angles



Sasaki’s Puzzles

Patterns to be made by folding a square of paper coloured on one side.

From

http://www.britishorigami.info/fun/sasakis-puzzles/