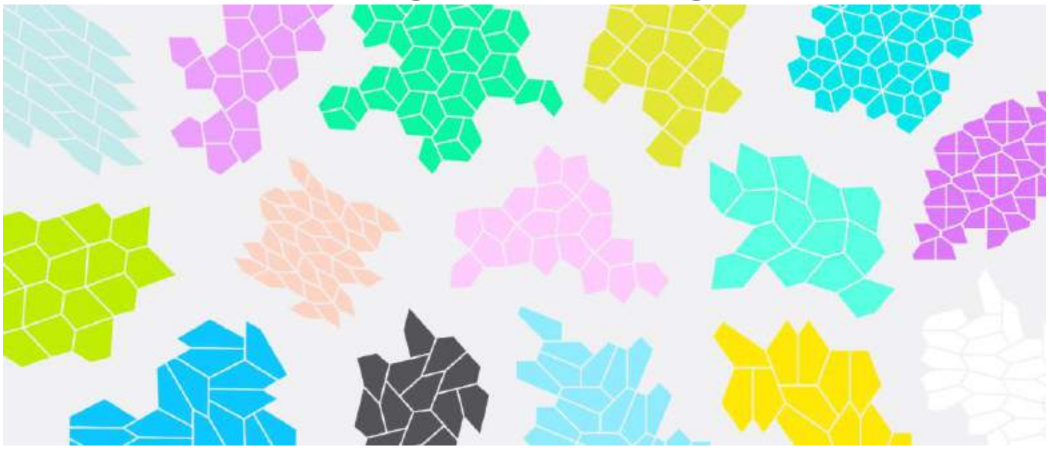
Pentagonal Tiling



Olena Shmahalo/Quanta Magazine
most others from Wikimedia Commons

David McQuillan
Thursday 1st February 2018

Picture shows examples of all 15 families of pentagonal tiling

Summary

- Tiling with a single convex polygonal tile
    - Three four and six sides.
    - A history of pentagonal tiling
    - Computer proof

- Related Topics
    - Penrose tiles
    - The Einstein problem

Tiling with convex polygons

Not talking about all tilings

Families of a single convex polygonal tile that can be used to tile the plane - monohedral convex tilings.

Non convex tilings includes images by Escher!

Any triangle, any quadrilateral, and three families of hexagonal tiles

All isohedral / tile transitive

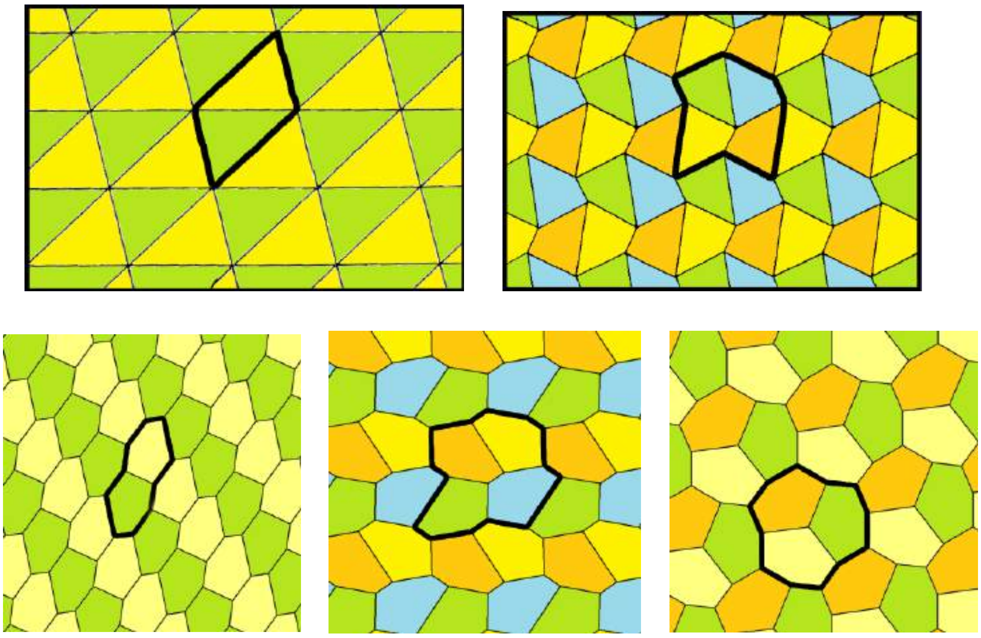
Tiling with polygons is an ancient problem and Kepler and Hilbert's 18th problem for instance both attacked the type of problem here – tiling the plane with congruent polygons.

The general problem of given a collection of tiles, possibly not convex, and finding if they can tile the plane is known to be undecidable.

R Berger 1966 The undecidability of the domino problem

R M Robinson 1971 Undecidability and nonperiodicity for tilings of the plane

3, 4, and 6 sided



Béla Bollobá

Filling the plane with congruent convex hexagons without overlapping

1963 Doctorate reviewed by Coxeter

All these tilings are isohedral / tile transitive – there is a congruence mapping any tile to any other.

Unsolved problem non-convex hexagons

1. B+C+D=360°, b=e
2. B+C+E=360°, b=e, d=f
3. B=D=F=120°, a=f, b=c, d=e

Impossible with convex heptagons

Just consider vertex tilings here

Sum of angles of heptagon is 900°

Average angle is 900°/7

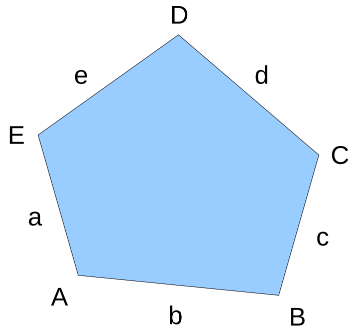
Therefore average corner has 360°/(900°/7) heptagons round it, i.e. 2.8 heptagons

Can't get this average without having some with only two which is a straight line

Not a full proof as doesn't include cases where corners meet on a side of a heptagon

1918 Karl Reinhart

Found 5 families of convex irregular pentagons tiling the plane



1. D+E = 180º
2. C+E = 180º, a=d
3. A = C = D = 120º, a=b, d=c+e
4. A = C = 90º, a=b, c=d
5. A = 60º, C = 120º, a=b, c=d

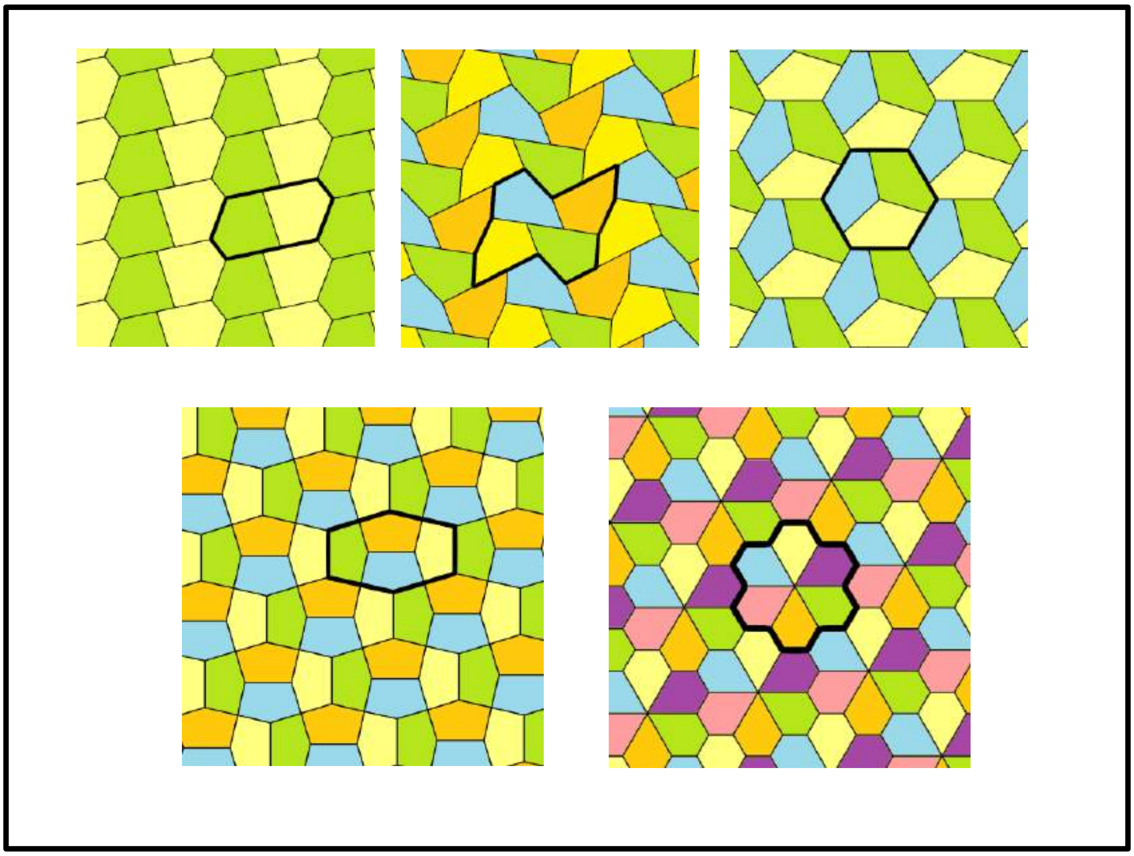
All these are isohedral

Sixteen different isohedral ways to tile type 1, mostly with extra constraints.

With an extra constraint a tile may satisfy the conditions of more than one family

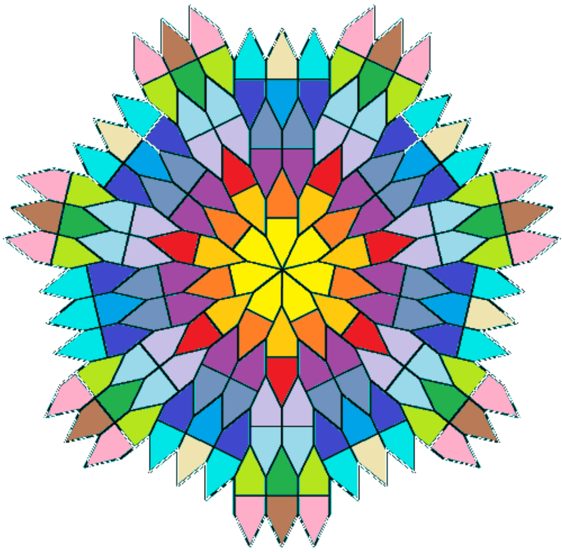
D+E=180°, D+B=180°, c=e, a=b+d

Satisfies the conditions for both type 1 and type 2 families



Emphasise the tilings may not be unique and there may be non-periodic tilings with the same tiles.

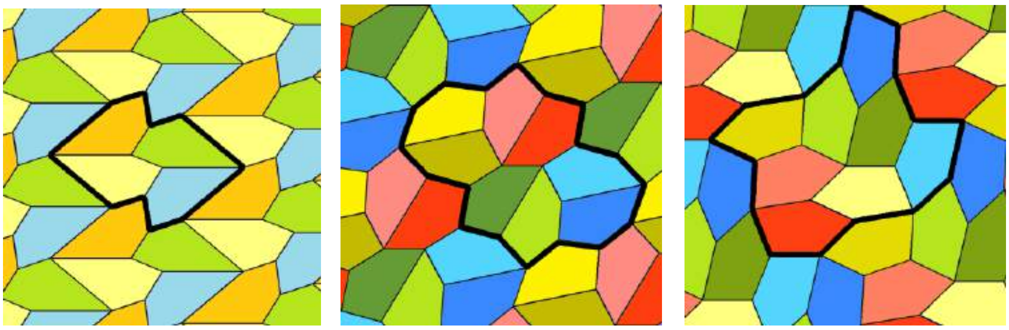
7-fold symmetry with a tile from family 1.



1968, Richard Kershner

No progress for 50 years, then Richard Kershner found three more families and thought he'd proved the list was complete – but didn't publish his proof because 'a complete proof would require a rather large book'.

Martin Gardiner wrote about this claim in his maths column in Scientific American in 1975 – which led to wide interest in the patterns among amateurs.



6. C+E=180°, A=2C, a=b=e, c=d
7. 2B+C=360°, 2D+A=360°, a=b=c=d
8. 2A+B=360°, 2D+C=360°, a=b=c=d

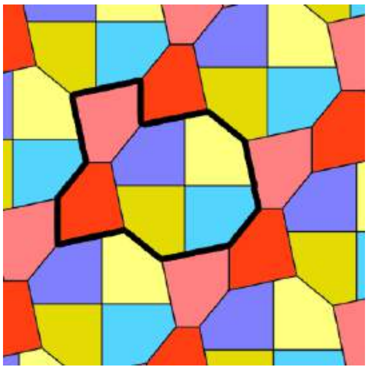
** Non-transitive tilings

1975, Richard E. James III

A computer programmer found one, now labelled 10 instead of 9.

10. E=90°,
 A+D=180°,
 A+2B=360°,
 a=e=b+d

** Has a vertex on a side



1976,1977 Marjorie Rice

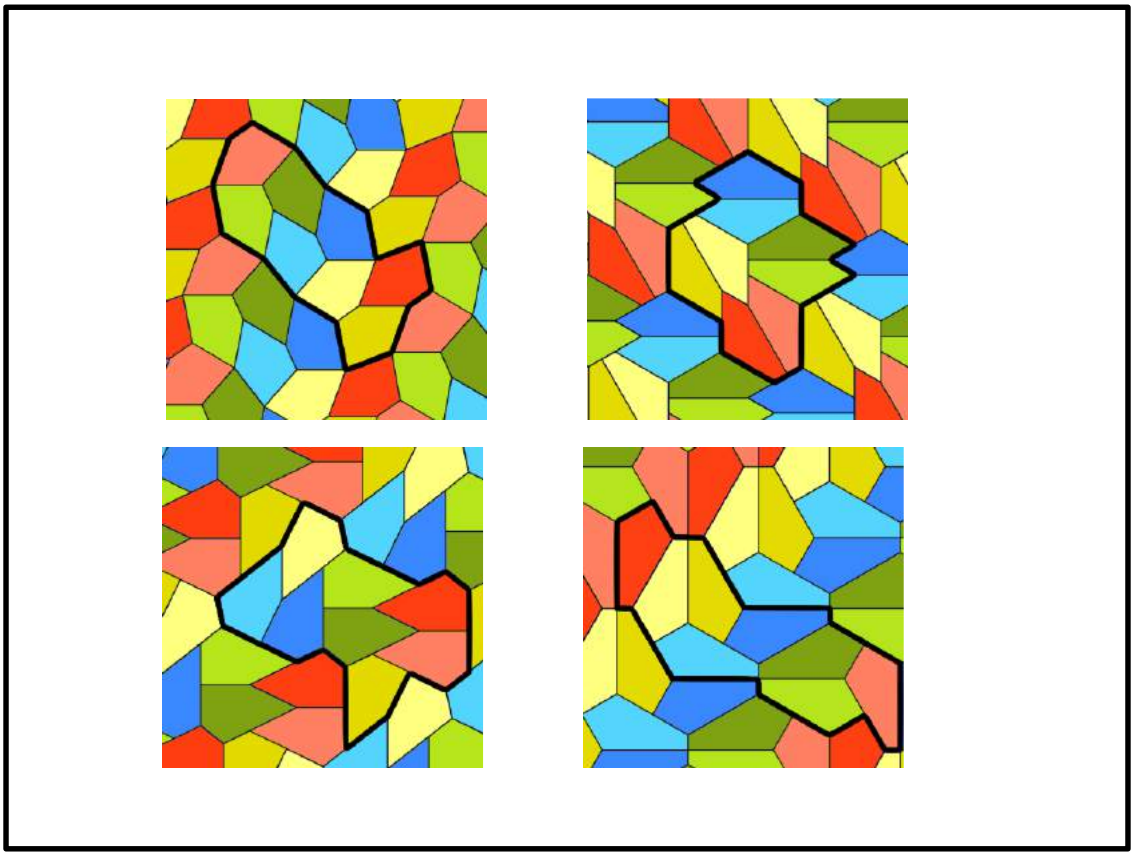
Amateur mathematician, found four new families.

9. $2A+C=D+2E=360°$, $b=c=d=e$

11. $A=90°$, $2B+C=360°$, $C+E=180°$,
    $2a+c=d=e$

12. $A=90°$, $2B+C=360°$, $C+E=180°$,
    $2a=d=c+e$

13. $B=E=90°$, $2A+D=360°$, $d=2a=2e$



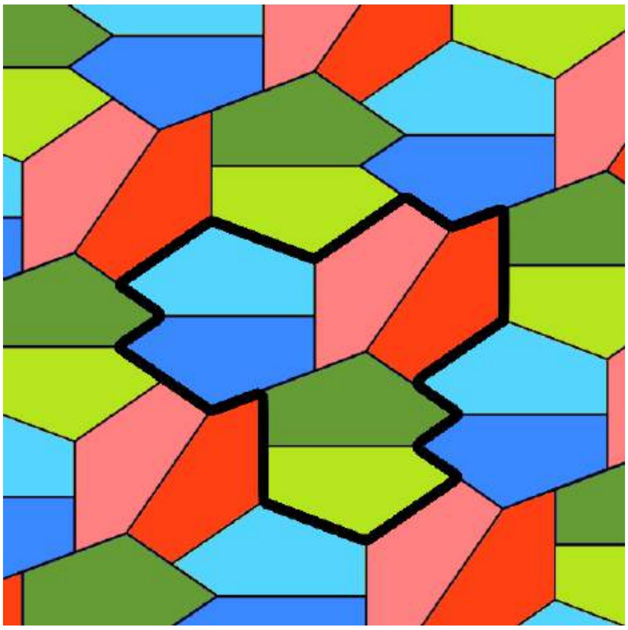
1985 Rolf Stein

** A completely determined tile not part of any other family.

14. A=90°, B≈145.34°, C≈69.32°

    D≈124.66°, E≈110.68°

    2a=2c=d=e, 2B+C=360°, C+E=180°



2015 Mann/McLoud/Von Derau

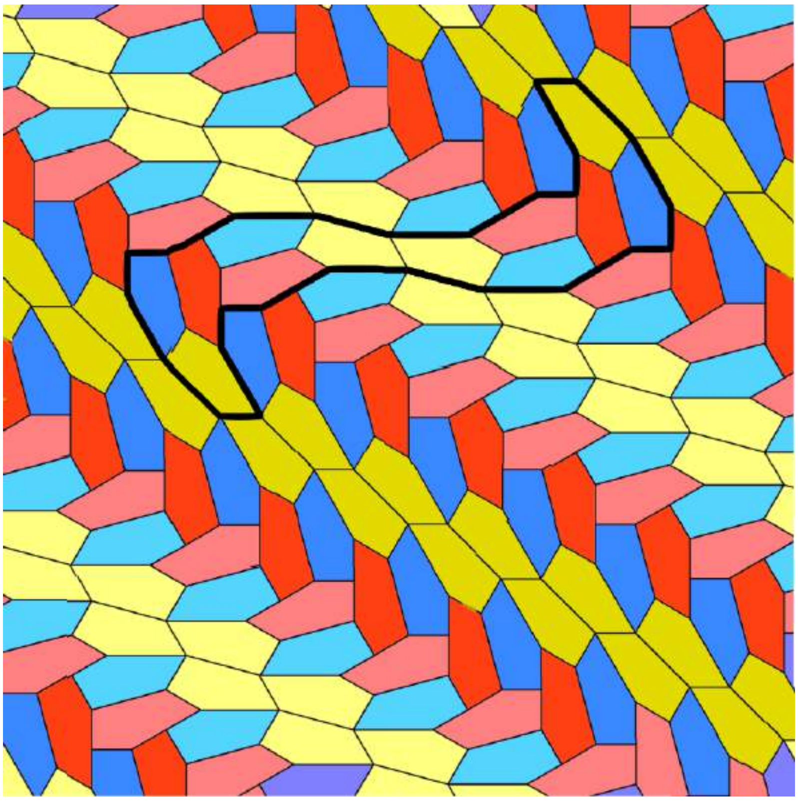
The 15th and final family of pentagonal tilings.

Like the 14^{th} tile it is completely determined.

Need 12 tiles to form a primitive unit that can tiled by translation.

15. A=150°, B=60°, C=135°, D=105°, E=90°

    a=c=e, b=2a, $d=a\sqrt{(2+\sqrt{3})}$



2017 Michaël Rao

Proved the list is complete

- Proof not yet fully checked.

Computer assisted proof rather similar to the 4-colour theorem.

Four colour theorem was proven in 1976 by Kenneth Appel and Wolfgang Haken. They found a set of 1936 maps which cannot be part of a smaller counterexample – then showed all these can be four coloured.

Preliminaries

Angles are a multiple of π so 2 is full circle.

A vertex type is a vector giving how many of each corner angle is used at a vertex.

(1,0,0,2,1) One of vertex 1, 2 of vertex 4 and 1 of vertex 5.

First showed that if a tiling could be done with a vertex type used only a vanishing amount overall then that vertex type is unnecessary for a full tiling.

Internal angles of pentagon add up to 3.

Corrected vector type has each entry doubled for a vertex on an edge so the total angles make a full circle.

Can find larger and larger circles each with a finite number of tilings containing previous ones and so have a path to infinity.

Opposite of vanishing amount is positive density. As a circle gets bigger and bigger the limit of the proportion of the vertex type in the interior is greater than zero.

Constraints on the angles

Next step was to set up some constrains on possible compatible combinations of vertex types.

Total angles of pentagon 3·π.
Total of angles at a vertex 2·π (or π but consider as doubled)

Biggest angle must be at least 3/5·π
Second biggest must be at least 1/2·π
Third biggest must be at least 1/3·π

Constraints are vectors in N^5 that when multiplied by the angles sum up to $2 \cdot \pi$

e.g. 11100, 00004 gives the constraints

$A+B+C=2 \cdot \pi$ and $4E=2 \cdot \pi$ or $2E=\pi$

Compattible combinations are ones that allow the sum of the angles to be $3 \cdot \pi$

* Some logic I'm not fully happy with yet * translates the constraints into equalities to zero or linear combinations that sum to zero.

Used a recursive program to show there are 371 of these compatible combinations.

Trying the possibilities

Then had a program try and do a tiling of a large area with each set. A fairly complex task!

Came up with 24 possible families.

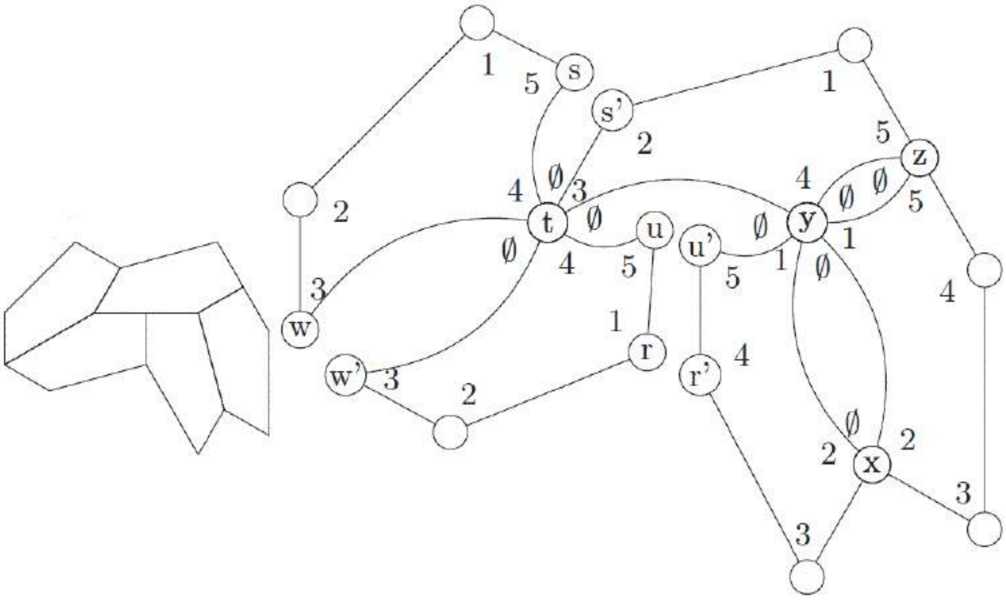
Four were special cases of others
Five were degenerate cases

The 15 already known are all there are.

Disappointed not to find one but was mainly developing the machinery to tackle the 'Einstein problem' we see later.

Mathematica was used for all the programming and everything ran in about 40 seconds – which is an age in modern computer terms!

Generating a tiling



Deals with completing edges and vertices.

A vertex is complete when it is equal to one of the allowed vertex types for the tiling.

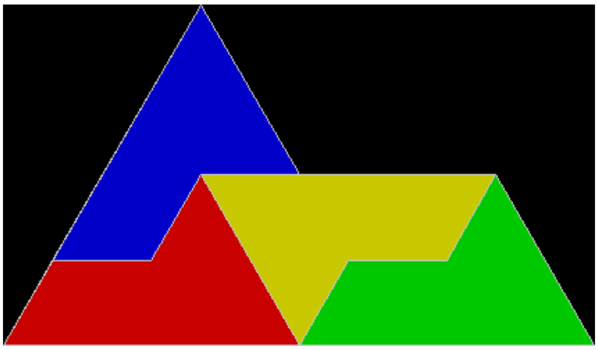
An edge can have a vertices along it where the angles are a half of an allowed vertex type.

Recursion involves matching up edges or else allowing an extra vertex on a side, or fitting an extra tile in a corner whilst still being a subset of an allowable vetex type.

Above when r and r' matched rather than an extra edge being put in it determines side 45 = 15

Sphinx Tile

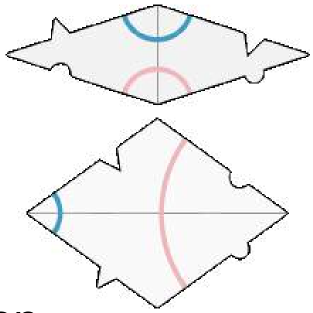
A non-convex pentagonal tile which is easy to tile regularly with but has a nice non-regular tesselation by 'deflation'.



This is a pentagonal 'rep-tile'. The replication is via sub-tiles of the same shape.

Penrose tiles

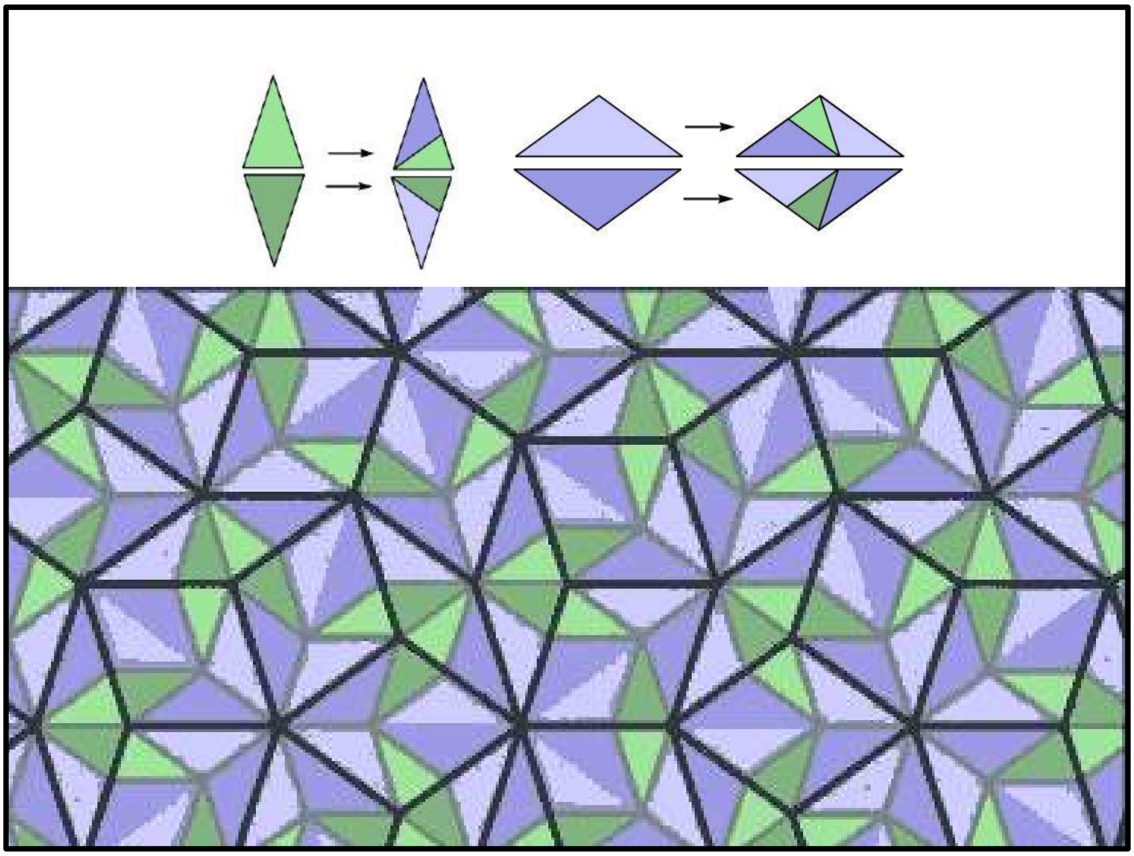
Two tiles that can tile the plane,
but not periodically.



Notches to ensure put together
properly.

Publicized by Martin Gardner
in January 1977 Scientific American

Inflation and deflation can be used to get more or less fine tilings



These images from

Austin, David (2005a), "Penrose Tiles Talk Across Miles", Feature Column, Providence: American Mathematical Society.

The following describes inflating and deflating and how a pattern can be expanded because of the strong constraints

Austin, David (2005b), "Penrose Tilings Tied up in Ribbons", Feature Column, Providence: American Mathematical Society.

http://www.ams.org/publicoutreach/feature-column/fcarc-ribbons

Einstein problem

Find a single tile that tiles the can tile the plane but not periodically.

Penrose tiles almost do this but there are two of them. They need notches put on the sides to ensure they only match in a particular way.

The Socolar–Taylor tile does it with hexagonal tiles – but the matching rule can only be done with disconnected pieces not notches.

Michaël Rao is developing his program in the hope of finding an einstein tile.

ein-stein = one tile