Kepler's laws of planetary motion

Based on

The Motions of Planets Around the Sun

by Richard Feynman

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Images from Wikipedia Commons or my own

And heavily changed by me!

Scheme

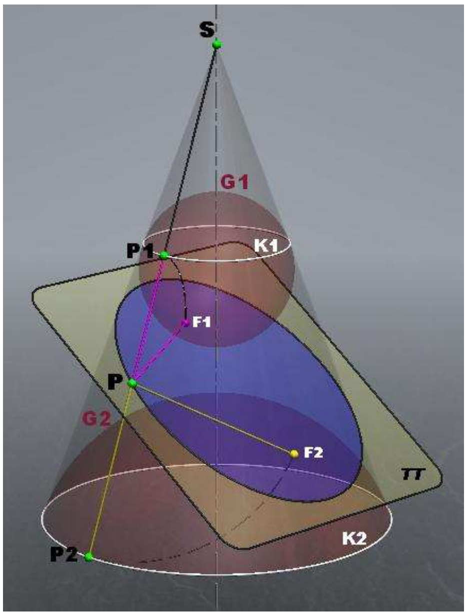
- A bit about ellipses
- A quick history
- Kepler's Laws – K1, K2, K3
- Newton's Laws
- Angular momentum - K2
- Universal gravitation – from K3 simplified
- Orbits are elliptical - K1
- Orbital periods - K3

Some properties of Ellipses

Ellipse can be drawn using a string around two focus points.

Shown here using Dandelin spheres in a cone

P F1 + P F2 =
  P P1+ P P2 =
  constant

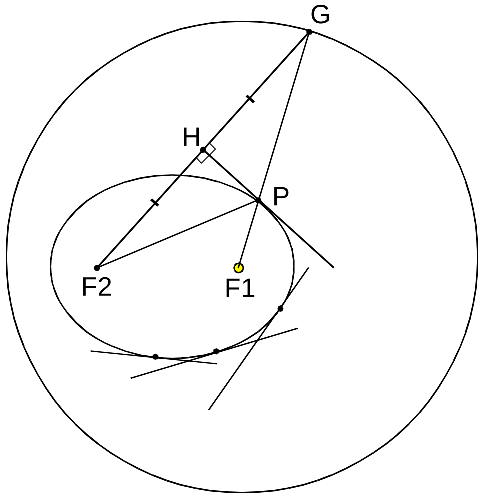


Construction using tangents

F_1 is the centre of the circle.

Tangents are perpendicular bisectors of F_2G

$F_2P + PF_1 =$
$GP + PF_1 = GF_1$
constant

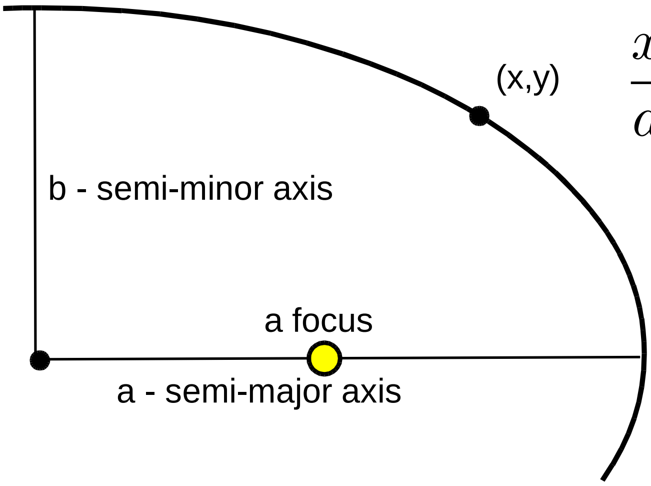


The ellipse is formed by all the tangents.
Must be tangents as any other points would be further out.

Also can see a light ray from F1 would get reflected by the ellipse to F2.

Cartesian co-ordinates

Consider as a squashed circle



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Quick History

- Tycho Brahe did observations which were five times better than those done previously. This required many more epicycles to fit the orbits.
- From the Copernican system with Earth circling the Sun Kepler came to the conclusion that since the Earth wasn’t perfect that the planets may not go round in perfect circles
- Took many years of work to work out his three Laws which fit the planets orbits.

Tycho Brahe Danish nobleman 4 December 1546 – 24 October 1601. astronomer, astrologer and alchemist,
https://en.wikipedia.org/wiki/Tycho_Brahe

Johannes Kepler December 27, 1571 - November 15, 1630 German astronomer, astrologer and mathematician
https://en.wikipedia.org/wiki/Johannes_Kepler

Kepler’s Laws

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Published first two laws in 1609 and the third in 1619. And they have been simplified a bit into this form.

Took a lot of work. Kepler quoted Vergil:

Galatea seeks me mischeviously, the lusty wench:
She flees the willows, but hopes I see her first

Newton’s Laws

i. Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed

ii. The alteration of motion is ever proportional to the motive force impress'd; and is made in the direction of the right line in which that force is impress'd

iii. To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Quite different from Kepler’s Laws

Kepler’s Laws are empirical, descriptive of a specific system,

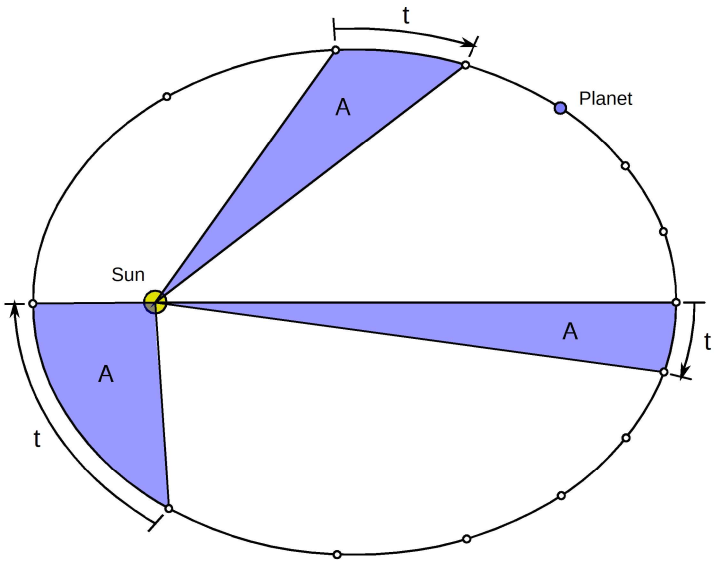
Newon’s are general – they can be applied to many other circumstances

Kepler’s Second Law

A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

Using Newton’s first and second laws this is equivalent to saying that any force on the planets must be directed directly to the sun so angular momentum is conserved. Any other direction and it would not be conserved.

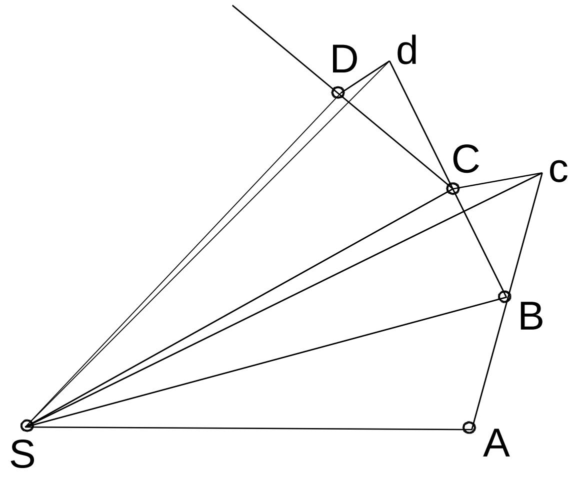
Kepler’s Second Law



Same time to traverse the three blue regions which are of equal area.

Angular Momentum

- Bc = AB
  Area SAB = SBC
- Impetus towards S applied at B
- Bc diverted to BC,
  Cc is parallel to SB
- Area SBC = SBc
- Area SBC = SAB



Newton used a diagram like this one so everything was geometrical rather than using his calculus.

Universal gravitation

"I deduced that the forces which keep the planets in their orbs must be reciprocally as the squares of their distances from the centres about which they revolve, and thereby compared the force requisite to keep the moon in her orb with the force of gravity at the surface of the earth and found them to answer pretty nearly."

Idea was in the air for a while before but Newton was the first to properly join the dots between gravitation on Earth and in the heavens

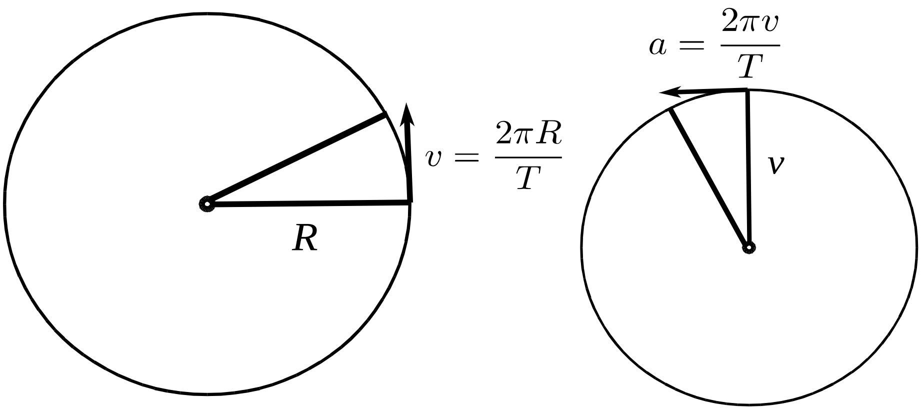
From Kepler’s Third Law

The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Using the special case of a circular orbit we can deduce Newton’s law of universal gravitation. Robert Hooke suggested an inverse square law but didn’t put in the masses or thought to apply it to apples or the tides on Earth.

Inverse Square Law

If a planet moves in a circular orbit
radius R and takes time T
with v = velocity and a = acceleration



The velocity diagram is also called a hodograph
https://en.wikipedia.org/wiki/Hodograph

A visual representation of a variable vector with one end fixed. Here all the velocity vectors have their base fixed.

Inverse Square Law continued

$$a = \frac{2\pi}{T} \cdot \frac{2\pi R}{T} = \frac{4\pi^2 R}{T^2}$$

$$T^2 = kR^3$$

From Kepler's Third Law

$$a = \frac{4\pi^2 R}{kR^3} = \frac{4\pi^2}{kR^2}$$

Gives Inverse square Law

$$F = a \cdot m$$

$$F = G\frac{Mm}{R^2}$$

Force is mass by acceleration – and this should hold for both the masses by Newtons third Law

Newton's third law used to justify having Mm in his formula, the Sun should be affected the same as the planets.

In fact for the Two body problem one should consider both bodies as going round a common centre of mass but for the solar system all the planets are much smaller than the Sun.

Keplers First Law

The orbit of a planet is an ellipse with the Sun at one of the two foci.

Newton's proof used arcane theorems from Apollonius' Conic sections working directly from that equal areas are swept out in equal times.

Feynman gave a much simpler proof using equal angles giving equal accelerations. In fact this is equivalent to something called the Laplace-Runge-Lenz vector being conserved, which keeps being forgotten and being independently found by loads of people.

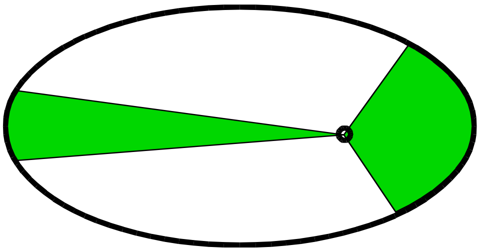
Laplace-Runge-Lenz vector
https://en.wikipedia.org/wiki/Laplace%E2%80%93Runge%E2%80%93Lenz_vector

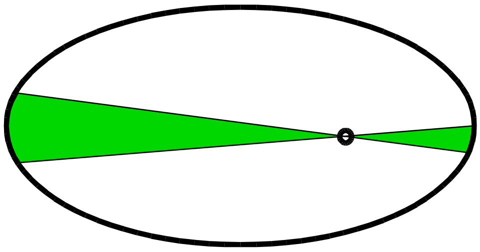
Or more simply the Lenz vector

Feynman didn't mention the Lenz vector

Different approaches

- Equal areas – equal times as in Kepler's second law
- Equal angles – equal velocity change





Newton - equal areas

Feynman - equal angles

Equal acceleration in equal angles

Since angular momentum is conserved the time to go round angle *θ* is proportional to $r^2\theta/2$ – the area of a narrow triangle. The acceleration inwards is GM/r^2, so the total change in velocity while going round angle *θ* is proportional to

$$\frac{r^2\theta}{2} \cdot \frac{GM}{r^2} = \left(\frac{GM}{2}\right)\theta$$

The change in velocity is proportional to the angle.

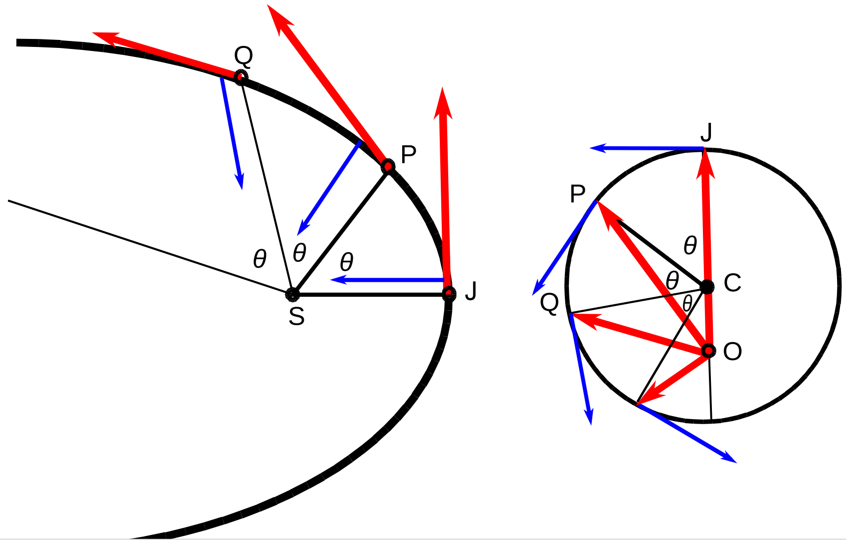
Change of velocity in equal angles

For constant changes in angle the magnitude of change of velocity is always equal and the direction is towards the Sun – i.e. along the radius vector. So the velocities in a velocity diagram form a circle.

But the origin for the velocities is not the origin of the circle.

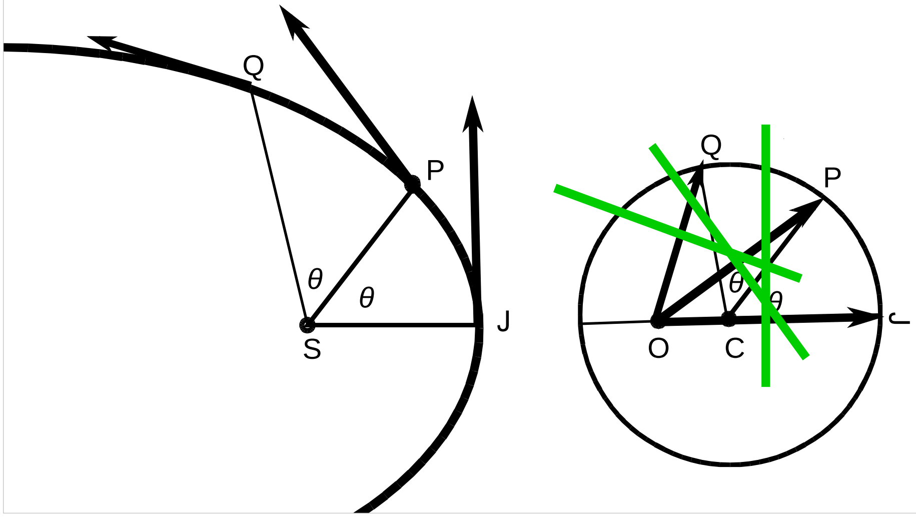
Velocity Diagram

Size of velocity change from J to P equals that from P to Q and angle changes by θ.



Feynman’s trick

Turn the velocity diagram sideways and draw bisectors of the velocities.

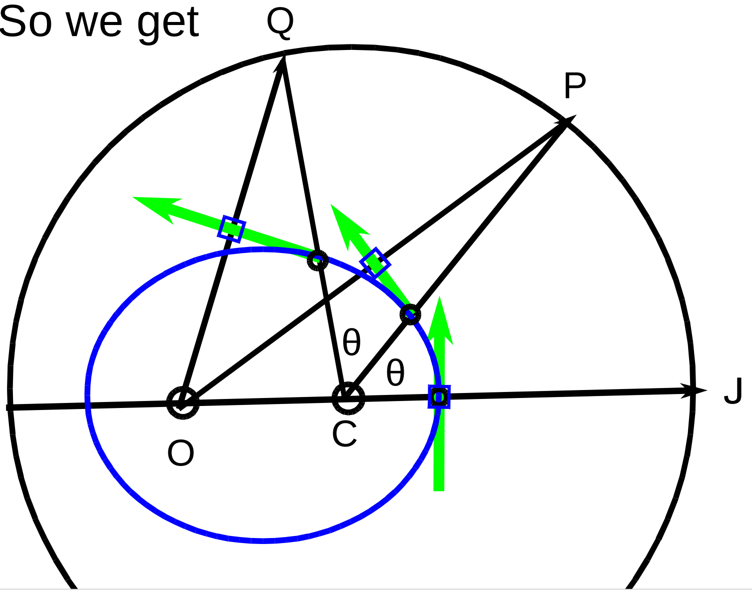


The green lines are in the same directions as the velocities and at the same angle.

So what is the figure one traces out if these green lines form an envelope with the tangents at the given angles?

Velocities are tangents of orbit

We know that the velocities are all tangents to the orbit. So we get



Start with the bisector on OJ which is the same point as where CJ meets it. Follow the bisector to meet CP. As the angle θ gets smaler it gets closer to where the bisector of OP meets CP. From there follow the bisector to meet CQ. As the angle θ gets smaller it gets closer to where the bisector of OQ meets CQ.

The ellipse is the limit of the shape as θ gets smaller and at each point the tangent is in the direction of the velocity. These angles and direction and positions are where the orbit goes and therefore the orbit is an ellipse. The only difference is the scale of the ellipse.

Et voilà

We have a figure which at the same angles as the planet from the Sun has the same tangents, and there can only be one such figure at a given scale. And that figure is an ellipse as we saw at the beginning.

So that is Kepler’s First Law derived.

Orbital Periods

- Kepler’s third law applied to circular orbits was used to infer the universal gravitational law.
- Now need to prove the general form of the law.

  Feynman ignored this bit and went on to other things instead.
- We need to calculate the area covered per unit time – then divide that into the area of the ellipse to give the period.

This is a maths group not a physics group! Can’t leave holes around the place.

Speed from path

If we see the path of an object under acceleration we can calculate the speed along and time from distance as follows

Distance along $d = vt$

Distance fallen $h = \frac{1}{2}kt^2$

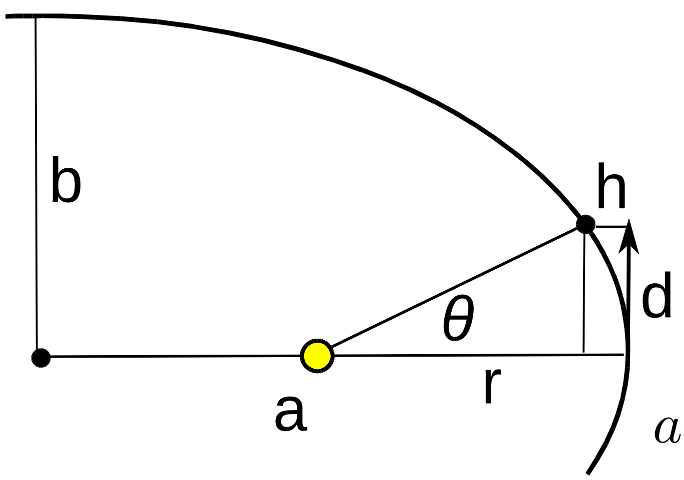
Time $t = \sqrt{\frac{2h}{k}}$

Speed along $v = \frac{d}{t}$

Normal business of the parabolic path of an object under acceleration

The variables

Approximately if θ is very small



$$d = vt$$

$$d \approx r\theta$$

$$h \approx \frac{1}{2}kt^2$$

$$k = \frac{GM}{r^2}$$

$$area \approx \frac{1}{2}r^2\theta$$

In this acceleration is to the left rather than down

a is semi-major axis
b is semi-minor axis

Area covered going round angle θ is approximately the same as a long triangle side r and angle θ.

Speed calculation

$$\frac{(a-h)^2}{a^2} + \frac{d^2}{b^2} = 1$$

$$a^2d^2 - 2ab^2h + b^2h^2 = 0$$

$$ad^2 \approx 2b^2h$$

$$h = \frac{ad^2}{2b^2} = \frac{1}{2}kt^2$$

$$t = \sqrt{\frac{2h}{k}} = \sqrt{\frac{ad^2r^2}{b^2GM}}$$

Approximating if h is small

Calculating the period

Sector area approx $\frac{1}{2}r^2\theta = \frac{1}{2}rd$

Per unit time $\frac{\frac{1}{2}rd}{\sqrt{\frac{ad^2r^2}{b^2GM}}} = \sqrt{\frac{GMb^2}{4a}}$

Area of ellipse πab

Period $T^2 = \frac{(\pi ab)^2}{\frac{GMb^2}{4a}} = \left(\frac{4\pi^2}{GM}\right)a^3$

Finite mass of the Sun

Haven't considered that the Sun has a finite mass and both the Sun and a planet go around a common centre of mass. Newton derived the correct form which makes a difference for the Earth and the Moon.

$$T^2 = \frac{4\pi^2}{G(M+m)}a^3$$

Easy enough using techniques like 'reduced mass' but not in Kepler's laws so left as an exercise :-)

Not too difficult, a number of different ways of doing this, for instance

Reduced mass
https://en.wikipedia.org/wiki/Reduced_mass