Stokes’ Theorem modernized

(with added hand waving)

The Gradient Theorem

$$\int_{l[\mathbf{p},\mathbf{q}]} \nabla f(\mathbf{r})\, d\mathbf{r} = f(\mathbf{r})\Big|_{\mathbf{p}}^{\mathbf{q}}$$

The integral of the grad of a function along a curve is equal to the difference of the function between the two ends.

Stokes' Theorem

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot dC$$

The integral of the curl of a vector field over a surface is equal to the line integral of the vector field along the perimeter.

The Divergence Theorem

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S \mathbf{F} \cdot d\mathbf{S}$$

The integral of the div of a vector field within a volume is equal to the integral of the field normal to the surface.

In Modern Form

$$\int_{\Omega} d\omega = \int_{\partial\omega} \omega$$

The integral of the exterior derivative of a differential form ω over an orientable manifold Ω is equal to the integral of the differential form over the boundary of the orientable manifold.

So what is a differential form?

$\omega = \mathbf{F} \cdot d\mathbf{r}$

$= f_x \, \mathrm{d}x + f_y \, \mathrm{d}y + f_z \, \mathrm{d}z$

This is a 1-form, a value for a line element. A function with no d's is a 0-form.

$$\sum a_{ij} \mathrm{d}x_i \wedge \mathrm{d}x_j$$

Is a 2-form or bivector giving the value for a surface element.

'∧' is the wedge product operator.

The Wedge Product

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{dx}$$
$$\mathrm{d}x \wedge \mathrm{d}x = 0$$

In 2 dimensions the space of surface element 2-forms $\mathrm{d}x \wedge \mathrm{d}y$ is one dimensional. In 3 dimensions the space of 2-forms is 3-dimensional.

Surface elements at a point in 3-dimensions can be considered as forming a 3 dimensional vector space.

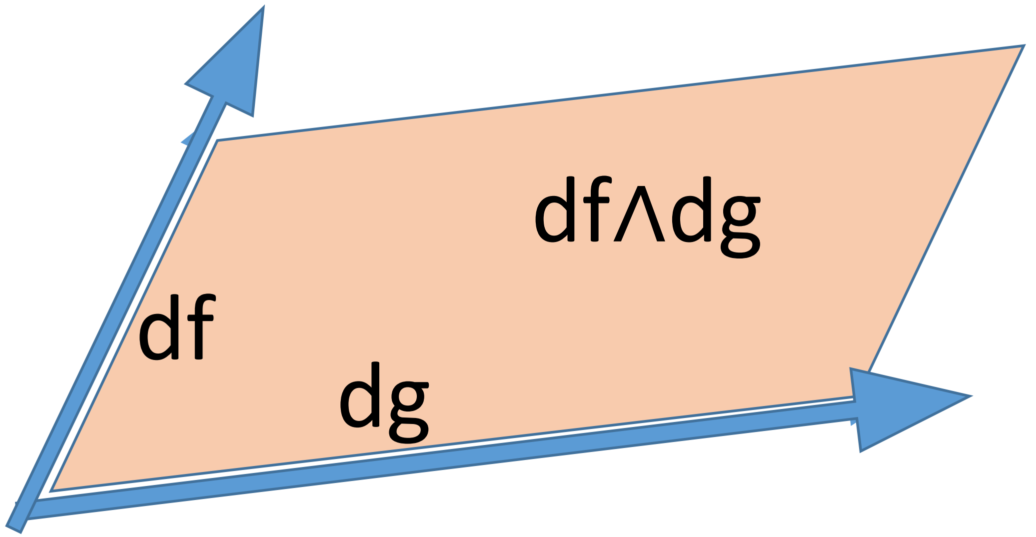
Calculating a surface element

$(f_x \, \mathrm{d}x + f_y \, \mathrm{d}y + f_z \, \mathrm{d}z) \wedge (g_x \, \mathrm{d}x + g_y \, \mathrm{d}y + g_z \, \mathrm{d}z)$

$= (f_x \, g_y - f_y \, g_x) \, \mathrm{d}x \wedge \mathrm{d}y +$

$(f_y \, g_z - f_z \, g_y) \, \mathrm{d}y \wedge \mathrm{d}z +$

$(f_z \, g_x - f_x \, g_z) \, \mathrm{d}z \wedge \mathrm{d}x$



Only the plane and area matters, not the shape

Exterior Derivative

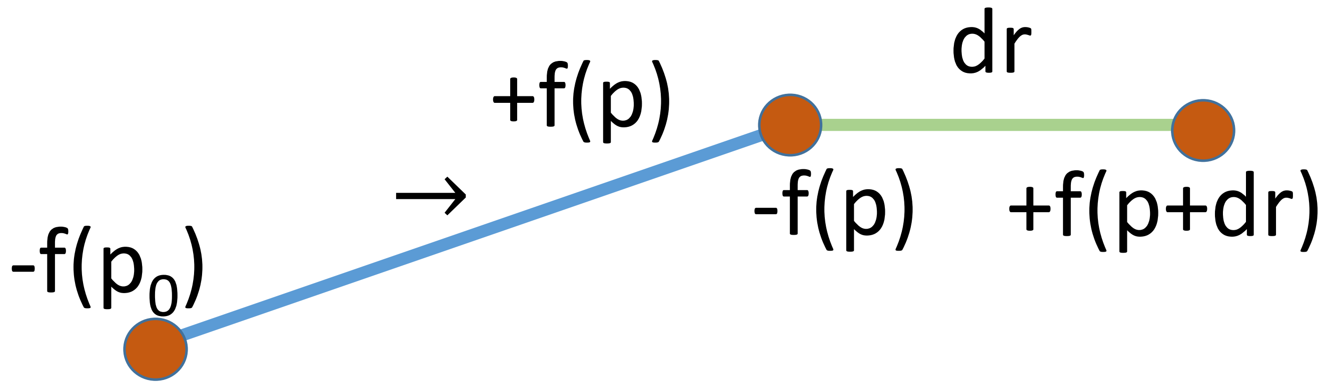
Given by

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$

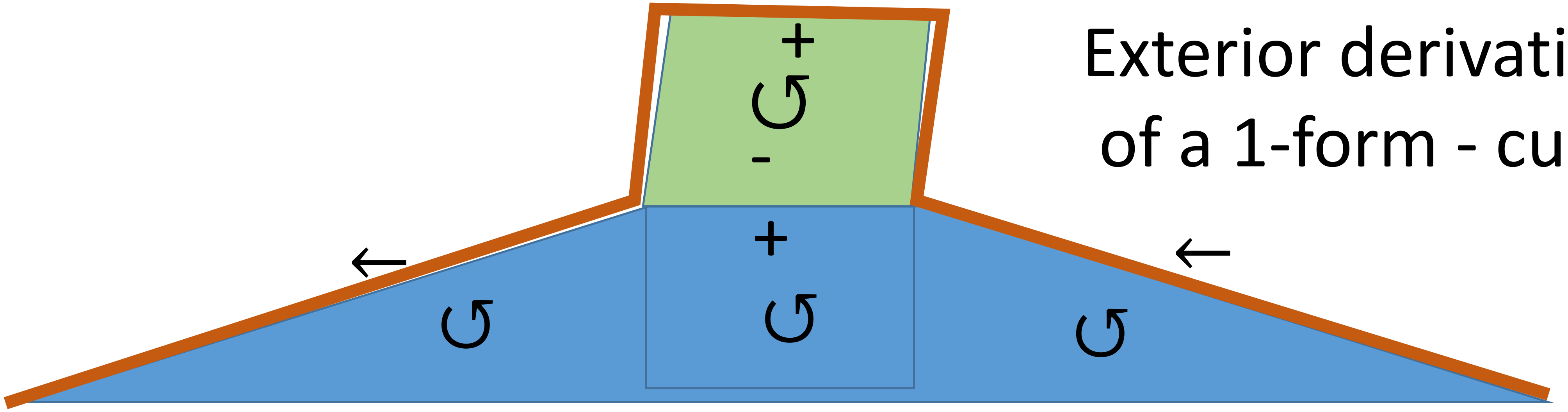
$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$$

Where α is a k-form.

Um uh - why is that useful?

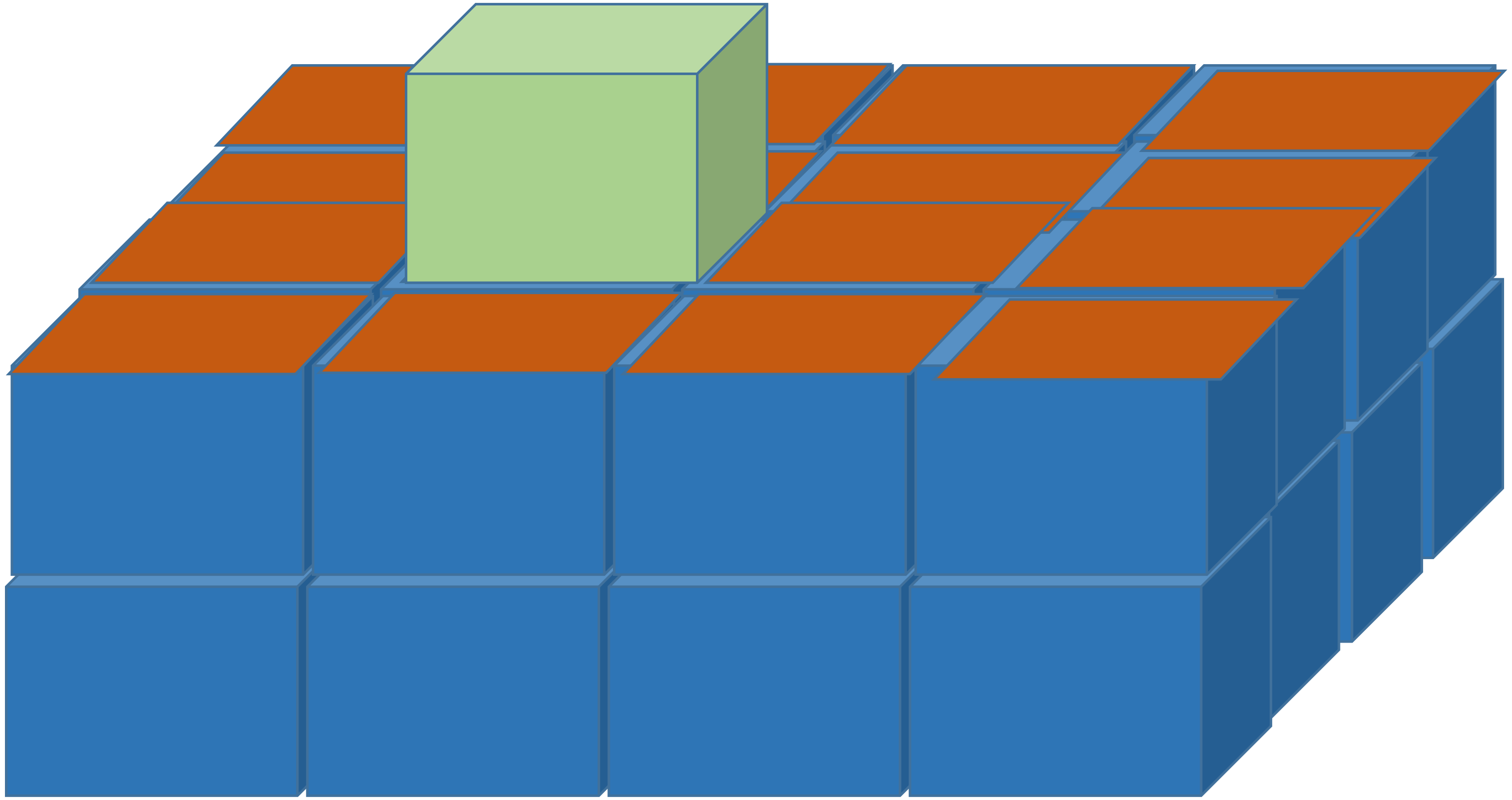


Exterior derivative of 0-form - grad



Exterior derivative of a 1-form - curl

And for the 2-form - div



Hodge Star changes k-form to n-k form

$$p \times q = \star\, p \wedge y$$

Type equation here.

$$\operatorname{grad} f = \nabla f = \mathrm{d}f$$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \star\, \mathrm{d}\mathbf{F}$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \star\, \mathrm{d} \star \mathbf{F}$$

$$\Delta f = \nabla^2 f = \star\, \mathrm{d} \star f$$